Problem 1.49

Evaluate the integral

$$J = \int_{\mathcal{V}} e^{-r} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau$$

(where $\mathcal{V}$ is a sphere of radius R, centered at the origin) by two different methods, as in Ex. 1.16.

Solution

Method 1

Use the result of part (b) of Problem 1.39.

$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta^3(\mathbf{r}).$$

Since the origin lies within the sphere of radius R centered at the origin, the integral is nonzero.

$$\begin{aligned} J = \int_{\mathcal{V}} e^{-r} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau &= \iiint_{x^2+y^2+z^2 \le R^2} e^{-\sqrt{\mathbf{r}\cdot\mathbf{r}}} \left[4\pi\delta^3(\mathbf{r}) \right] d\mathbf{r} \\ &= 4\pi \iiint_{x^2+y^2+z^2 \le R^2} e^{-\sqrt{\mathbf{r}\cdot\mathbf{r}}} \delta^3(\mathbf{r})\, d\mathbf{r} \\ &= 4\pi \left(e^{-\sqrt{\mathbf{0}\cdot\mathbf{0}}} \right) \\ &= 4\pi \end{aligned}$$

Method 2

Use Identity 5 inside the front cover to rewrite the integrand.

$$\begin{aligned} J = \int_{\mathcal{V}} e^{-r} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau &= \iiint_{x^2+y^2+z^2 \le R^2} e^{-r} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) dV \\ &= \iiint_{x^2+y^2+z^2 \le R^2} \left[\nabla \cdot \left(e^{-r} \frac{\hat{\mathbf{r}}}{r^2} \right) - \frac{\hat{\mathbf{r}}}{r^2} \cdot (\nabla e^{-r}) \right] dV \\ &= \iiint_{x^2+y^2+z^2 \le R^2} \nabla \cdot \left(e^{-r} \frac{\hat{\mathbf{r}}}{r^2} \right) dV - \iiint_{x^2+y^2+z^2 \le R^2} \frac{\hat{\mathbf{r}}}{r^2} \cdot (\nabla e^{-r})\, dV \\ &= \oiint_{x^2+y^2+z^2 = R^2} e^{-r} \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{S} - \iiint_{x^2+y^2+z^2 \le R^2} \frac{\hat{\mathbf{r}}}{r^2} \cdot (-e^{-r}\hat{\mathbf{r}})\, dV \\ &= \oiint_{r^2 = R^2} e^{-r} \frac{\hat{\mathbf{r}}}{r^2} \cdot (\hat{\mathbf{r}} R^2 \sin\theta\, d\phi\, d\theta) + \iiint_{r^2 \le R^2} \frac{e^{-r}}{r^2} (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})(r^2 \sin\theta\, dr\, d\phi\, d\theta) \end{aligned}$$

Continue the simplification.

$$\begin{aligned}
J &= \oiint_{r=R} \frac{e^{-r}}{r^2}(R^2 \sin\theta \, d\phi \, d\theta) + \iiint_{r\leq R} e^{-r} \sin\theta \, dr \, d\phi \, d\theta \\
&= \int_0^{\pi} \int_0^{2\pi} e^{-R} \sin\theta \, d\phi \, d\theta + \int_0^{\pi} \int_0^{2\pi} \int_0^{R} e^{-r} \sin\theta \, dr \, d\phi \, d\theta \\
&= e^{-R} \left(\int_0^{\pi} \sin\theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right) + \left(\int_0^{R} e^{-r} \, dr \right) \left(\int_0^{\pi} \sin\theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right) \\
&= e^{-R}(2)(2\pi) + (1 - e^{-R})(2)(2\pi) \\
&= 4\pi
\end{aligned}$$